2.3 Principle of Inclusion and Exclusion

The following result is well known and hence we omit the proof.

Theorem 2.3.1. *Let U be a finite set. Suppose A and B are two subsets of U. Then the number of elements of U that are neither in A nor in B are*

$$|U| - \big(|A| + |B| - |A \cap B|\big).$$

Or equivalently, $|A \cup B| = |A| + |B| - |A \cap B|$.

A generalization of this to three subsets A, B and C is also well known. To get a result that generalizes Theorem 2.3.1 for n subsets $A_1, A_2, \ldots, A_n$, we need the following notations:

$$S_1 = \sum_{i=1}^{n} |A_i|, \;\; S_2 = \sum_{1 \le i < j \le n} |A_i \cap A_j|, \;\; S_3 = \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k|, \cdots, S_n = |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

With the notations as defined above, we have the following theorem, called the inclusion-exclusion principle. This theorem can be easily proven using the principle of mathematical induction. But we give a separate proof for better understanding.

Theorem 2.3.2. *[Inclusion-Exclusion Principle] Let $A_1, A_2, \ldots, A_n$ be n subsets of a finite set U. Then the number of elements of U that are in none of $A_1, A_2, \ldots, A_n$ is given by*

$$|U| - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n. \tag{2.1}$$

Or equivalently,

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = S_1 - S_2 + \cdots + (-1)^{n-1} S_n. \tag{2.2}$$

Proof. We show that if an element $x \in U$ belongs to exactly k of the subsets $A_1, A_2, \ldots, A_n$, for some $k \geq 1$, then its contribution in (2.1) is zero. Suppose x belongs to exactly k subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$. Then we observe the following:

1. The contribution of x in $|U|$ is 1.

2. The contribution of x in S_1 is k as $x \in A_{i_j}$, $1 \le j \le k$.

3. The contribution of x in S_2 is $\binom{k}{2}$ as $x \in A_{i_j} \cap A_{i_l}$, $1 \le j < l \le k$.

4. The contribution of x in S_3 is $\binom{k}{3}$ as $x \in A_{i_j} \cap A_{i_l} \cap A_{i_m}$, $1 \le j < l < m \le k$.

   Proceeding this way, we have

5. The contribution of x in S_k is $\binom{k}{k} = 1$, and

6. The contribution of x in S_ℓ for $\ell \geq k+1$ is 0.

So, the contribution of x in (2.1) is

$$1-k+\binom{k}{2}-\binom{k}{3}+\cdots+(-1)^{k-1}\binom{k}{k-1}+(-1)^k\binom{k}{k}+0\cdots+0=(1-1)^k=0.$$

This completes the proof of the theorem. The readers are advised to prove the equivalent condition. ■

Example 2.3.3. 1. *Determine the number of 10-letter words using ENGLISH alphabets that does not contain all the vowels.*

Solution: *Let U be the set consisting of all the 10-letters words using ENGLISH alphabets and let A_α be a subset of U that does not contain the letter α. Then we need to compute*

$$|A_a \cup A_e \cup A_i \cup A_o \cup A_u| = S_1 - S_2 + S_3 - S_4 + S_5,$$

where $S_1 = \sum\limits_{\alpha\in\{a,e,i,o,u\}} |A_\alpha| = \binom{5}{1}25^{10}$, $S_2 = \binom{5}{2}24^{10}$, $S_3 = \binom{5}{3}23^{10}$, $S_4 = \binom{5}{4}22^{10}$ *and* $S_5 = 21^{10}$. *So, the required answer is* $\sum\limits_{k=1}^{5}(-1)^{k-1}\binom{5}{k}(26-k)^{10}$.

2. *Determine the number of integers between 1 and 1000 that are coprime to 2, 3, 11 and 13.*

Solution: *Let* $U = \{1,2,3,\ldots,1000\}$ *and let* $A_i = \{n \in U : i \text{ divides } n\}$, *for* $i = 2, 3, 11, 13$. *Then note that we need the value of* $|U| - |A_2 \cup A_3 \cup A_{11} \cup A_{13}|$. *Observe that*

$$\begin{aligned}
|A_2| = \lfloor\frac{1000}{2}\rfloor = 500,\ |A_3| = \lfloor\frac{1000}{3}\rfloor = 333,\ |A_{11}| = \lfloor\frac{1000}{11}\rfloor = 90,\ |A_{13}| = \lfloor\frac{1000}{13}\rfloor = 76,\\
|A_2\cap A_3| = \lfloor\frac{1000}{6}\rfloor = 166,\ |A_2\cap A_{11}| = \lfloor\frac{1000}{22}\rfloor = 45,\ |A_2\cap A_{13}| = \lfloor\frac{1000}{26}\rfloor = 38,\\
|A_3\cap A_{11}| = \lfloor\frac{1000}{33}\rfloor = 30,\ |A_3\cap A_{13}| = \lfloor\frac{1000}{39}\rfloor = 25,\ |A_{11}\cap A_{13}| = \lfloor\frac{1000}{143}\rfloor = 6,\\
|A_2\cap A_3\cap A_{11}| = 15,\ |A_2\cap A_3\cap A_{13}| = 12,\ |A_2\cap A_{11}\cap A_{13}| = 3,\\
|A_3\cap A_{11}\cap A_{13}| = 2,\ |A_2\cap A_3\cap A_{11}\cap A_{13}| = 1.
\end{aligned}$$

Thus, the required number is

$$1000-\big((500+333+90+76)-(166+45+38+30+25+6)-(15+12+3+2)-1\big) = 1000-720 = 280.$$

3. *(Euler's ϕ-function Or Euler's totient function) Let n denote a positive integer. Then the Euler ϕ-function is defined by*

$$\phi(n) = \big|\{k\ : 1 \le k \le n,\ \gcd(n,k) = 1\}\big|. \tag{2.3}$$

Determine a formula for $\phi(n)$ in terms of its prime factors.

Solution: *Let* $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ *be the unique decomposition of n as product of distinct* *primes* $p_1, p_2, \ldots, p_k$, $U = \{1, 2, \ldots, n\}$ *and let* $A_{p_i} = \{m \in U : p_i$ *divides* $m\}$, *for* $1 \leq i \leq k$. *Then, by definition*

$$\begin{aligned} \phi(n) &= |U| - S_1 + S_2 - S_3 + \cdots + (-1)^k S_k \\ &= n - \sum_{i=1}^{k} \frac{n}{p_i} + \sum_{1 \leq i < j \leq k} \frac{n}{p_i p_j} - \cdots + (-1)^k \frac{n}{p_1 p_2 \cdots p_k} \\ &= n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right). \end{aligned} \tag{2.4}$$